Important: Don't use arctan or arcsin ...
to solve for angles on the unit circle.

OK - to use arctan when you have a right triangle.

Example: Write $z = -\sqrt{3} - i$ in polar form.

Bad solution: $x = -\sqrt{3},\ y = -1$

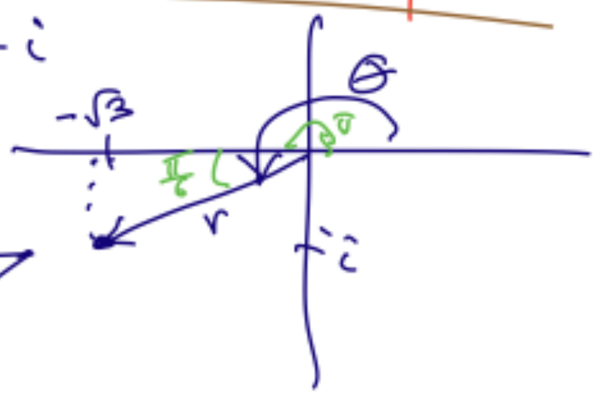
$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

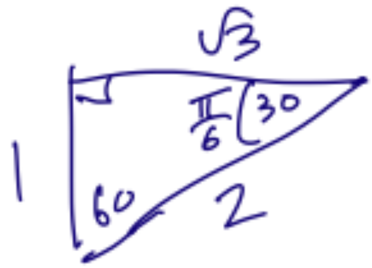
$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$

Solution: $z = 2e^{i\pi/6}$.

WRONG.

Correct solution $z = -\sqrt{3} - i$





$\therefore\ r = 2,\ \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}.$

Ans $\boxed{z = 2e^{i\frac{7\pi}{6}}}$

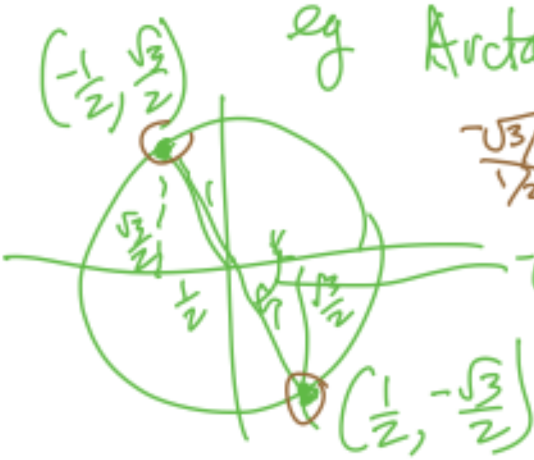
Principal Angles of "Inverse" Trig fcns.

Arctan(w) = principle angle whose tangent is w

Capital means "principle angle"

eg. $\boxed{-\frac{\pi}{2} < \text{Arctan}(w) < \frac{\pi}{2}}$

eg $\text{Arctan}(-\sqrt{3}) = -\frac{\pi}{3}$.



$\frac{-\sqrt{3}/2}{1/2}$ or $\frac{\sqrt{3}/2}{-1/2}$ $\quad \frac{y}{x}$

$-60° = \frac{\pi}{3}$

The angle $\frac{2\pi}{3}$ would make sense, cause $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$.

$\underline{\text{arctan}(-\sqrt{3})}$ = all possible angles θ such that $\tan(\theta) = -\sqrt{3}$.

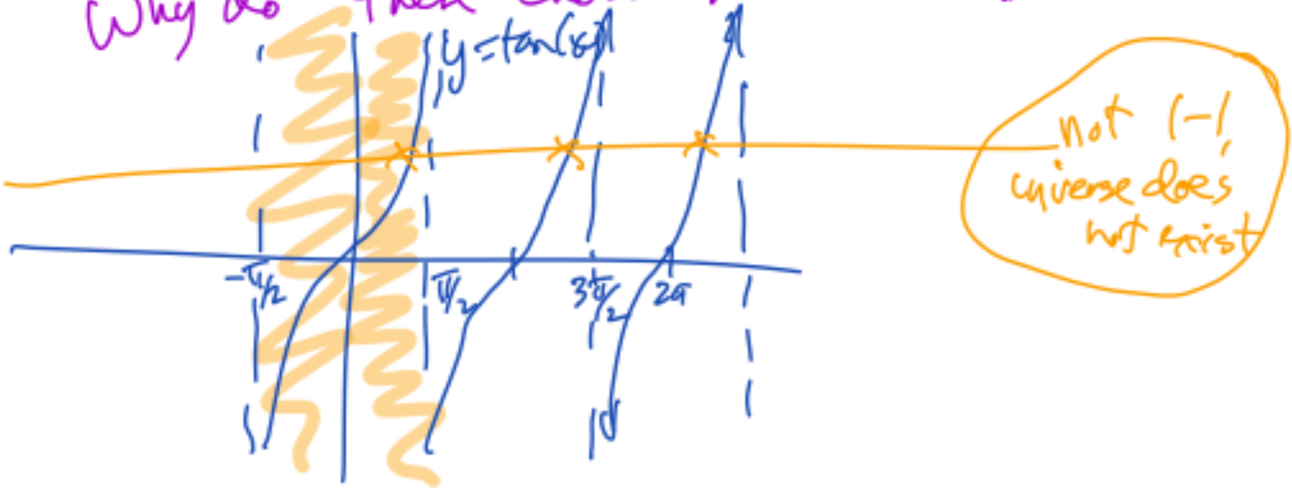
$$= \left\{\frac{2\pi}{3} + 2k\pi \;\middle|\; k \in \mathbb{Z}\right\} \cup \left\{-\frac{\pi}{3} + 2k\pi \;\middle|\; k \in \mathbb{Z}\right\}.$$

Principal Angles

$$-\frac{\pi}{2} \le \begin{matrix}\text{Arcsin}(\cdot) \\ \text{Arctan}(\cdot) \\ \text{Arccsc}(\cdot)\end{matrix} \le \frac{\pi}{2}$$

$$0 \le \begin{matrix}\text{Arccos}(\cdot) \\ \text{Arccot}(\cdot) \\ \text{Arcsec}(\cdot)\end{matrix} \le \pi$$

Why do these choices make sense?



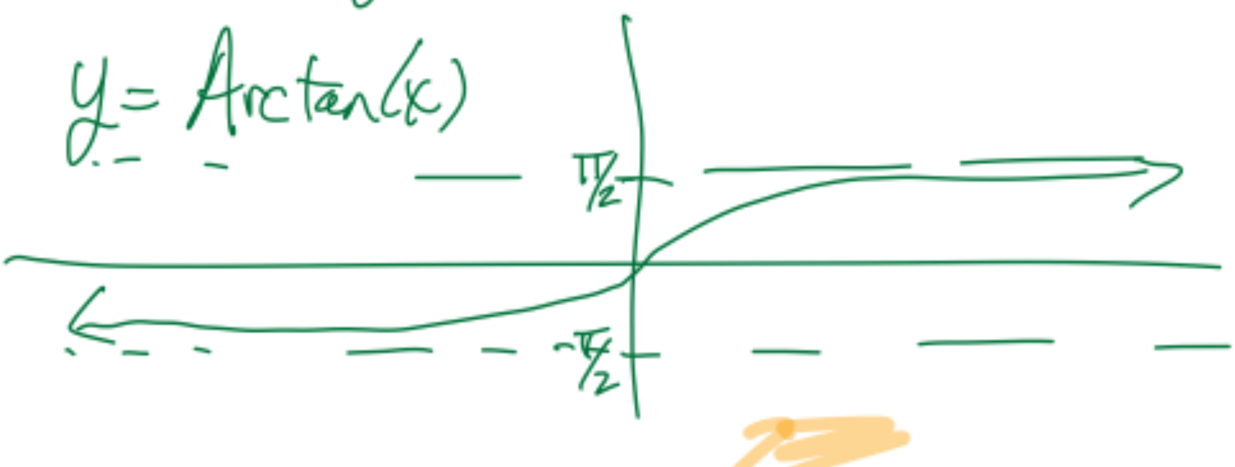
A function $f : \mathbb{R} \to \mathbb{R}$ is 1-1 if
whenever $f(x) = f(y)$, we have $x = y$.

$\iff$ for each single value of y,
there is only one value of x s.t. $y = f(x)$

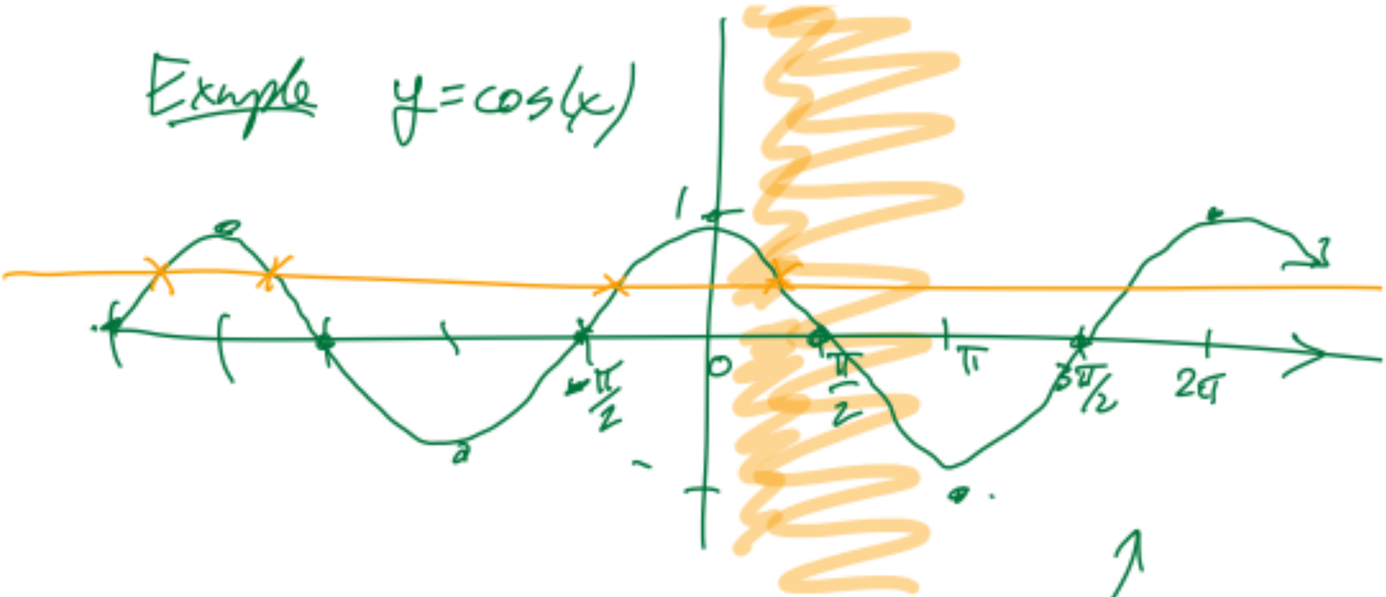
$\iff$ Each horiz line hits $y = f(x)$ at
most one time.

$\iff$ $\exists$ inverse fn g to f.
$g(f(x)) = x$, $f(g(x)) = x$.

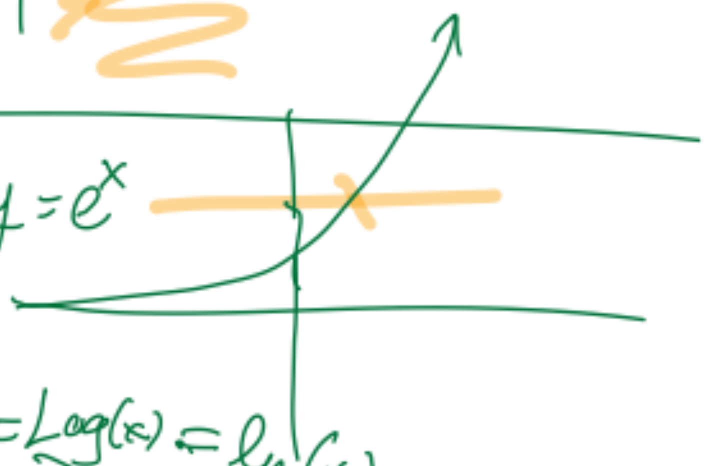
$y = \text{Arctan}(x)$



Example $y=\cos(x)$



Real variables $y=e^x$



$\Rightarrow$ well-defined inverse $y=\mathrm{Log}(x)=\ln(x)$

$$\mathrm{Log}(e^x)=x$$

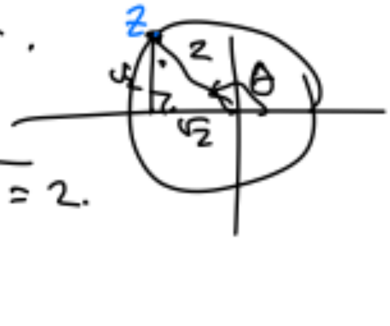
$$e^{\mathrm{Log}\,x}=x.$$

With complex #s — we don't just get one answer.

Interlude: Principal Argument $\mathrm{Arg}(z)$

$$-\pi<\mathrm{Arg}(z)\le\pi.$$

$z=-\sqrt{2}+\sqrt{2}\,i$ $\quad |z|=\sqrt{2+2}=2.$



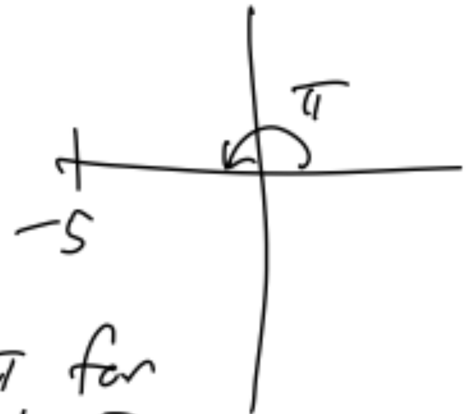
$$\operatorname{Arg}(z) = \frac{3\pi}{4}.$$

$$\arg(z) = \frac{3\pi}{4} + 2k\pi \text{ for } k \in \mathbb{Z}.$$

infinite # of answers.

multivalued fcn.

$$\operatorname{Arg}(-5) = \pi$$



$$\arg(-5) = \pi + 2k\pi \text{ for } k \in \mathbb{Z}$$

$$= (2k+1)\pi \text{ for } k \in \mathbb{Z}.$$

Let's try to find the inverse of

$$e^z = e^x e^{iy}$$

$w = e^z$ Want to write z as a fcn of w.

$$|w| = |e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x|$$

$$= e^x \text{ real.}$$

$$\Rightarrow |w| = e^x \Rightarrow x = \underset{\text{natural log.}}{\operatorname{Log}}(|w|) \qquad |w| \neq 0.$$

I have found the x part of $z =$ fcn of ω.

$$\arg(\omega) = \arg(e^{z}) = \arg(e^{x}e^{iy})$$

$$= \arg(e^{iy}) = y \qquad \arg(Re^{i\theta}) = \theta + 2k\pi.$$

$\therefore$ $z = x + iy$

$$z = \operatorname{Log}(\omega)$$

$$z = x \quad + iy$$

$$z = \underbrace{\operatorname{Log}(|w|) + i\operatorname{Arg}(\omega)}_{\text{inverse of } e^{z}}.$$

In summary: e^{z} is not 1-1 on the complex plane.

principal log

$$\boxed{\begin{aligned} \operatorname{Log}(z) &= \operatorname{Log}(|z|) + i\operatorname{Arg}(z) \\ \log(z) &= \underbrace{\operatorname{Log}(|z|)}_{\text{natural log}} + i\arg(z). \end{aligned}}$$

multivalued log

log of a complex #.

eg $\mathrm{Log}(-5) = \mathrm{Log}(|-5|) + i\,\mathrm{Arg}(-5)$

$= \mathrm{Log}(5) + i\pi$

$\log(-5) = \mathrm{Log}(5) + i(\pi + 2k\pi),\quad k \in \mathbb{Z}.$

$e^{\log(-5)} = -5$

$e^{(\mathrm{Log}(5) + i\pi)} = e^{\mathrm{Log}(5)} e^{i\pi} = 5(-1) = -5.$

Example $\mathrm{Log}(-\sqrt{2} - \sqrt{2}i)$

$= \mathrm{Log}(|-\sqrt{2} - \sqrt{2}i|) + i\,\mathrm{Arg}(-\sqrt{2} - \sqrt{2}i)$

$\underbrace{\sqrt{\underbrace{(\sqrt{2})^2}_{2} + \underbrace{(\sqrt{2})^2}_{2}}}_{2}$

$\mathrm{Log}(\sqrt{2} - \sqrt{2}i) = \mathrm{Log}(2) + i\left(-\frac{3\pi}{4}\right)$

$\mathrm{Log}(\overset{+}{\text{real}\#}) = \text{ordinary natural log}\left(\overset{+}{\text{real}\ \#}\right).$

We can check $e^{\text{Log}(2)+i\left(\frac{-3\pi}{4}\right)} = -\sqrt{2}-\sqrt{2}i$. ✓

<u>Exercise</u>: Find all possible values of

$$(2i)^{3i-2} = \left(e^{\log(2i)}\right)^{(3i-2)}$$

$$\log(2i) = \text{Log}|2i| + i\arg(2i)$$
$$= \text{Log}|2| + i\left(\frac{\pi}{2}+2k\pi\right)$$

for $k\in\mathbb{Z}$.

$$(2i)^{3i-2} = \left(e^{\left(\text{Log}|2|+i\left(\frac{\pi}{2}+2k\pi\right)\right)(3i-2)}\right)$$

for $k\in\mathbb{Z}$.

$$= e^{-2\text{Log}(2) - \frac{3\pi}{2} - 6k\pi + i(\pi+4k\pi) + i(3\cdot\text{Log}(2))}$$

$$= e^{-2\text{Log}(2)-\frac{3\pi}{2}-6k\pi}\,\underbrace{e^{i(\pi+4k\pi)}}_{=-1}\,e^{i(3\text{Log}(2))}$$

$$= \boxed{-e^{-2\text{Log}(2)-\frac{3\pi}{2}-6k\pi}\,e^{i(3\text{Log}\,2)}}$$

$= \text{all possible values of } (2i)^{(3i-2)}$

$$= -\underbrace{e^{-2\log(2)}}\, e^{-\frac{3\pi}{2}-6k\pi}\, e^{i(3\log(2))}$$

$$= \boxed{-2^{-2}\, e^{-\frac{3\pi}{2}-6k\pi}\, e^{i(3\log(2))}}$$

Note: This also happens if we take roots.

$$\sqrt{2i+7} = (2i+7)^{1/2}$$

$$= e^{\log(2i+7)\frac{1}{2}} \;\ldots\ldots$$

all possible values of $\sqrt{2i+7}$

← principal value of $\sqrt{2i+7}$.

BTW → we can compute all values another way.

polar

$$2i+7 = \underbrace{\sqrt{53}}_{r}\, e^{i\,\mathrm{Arctan}\left(\frac{2}{7}\right)}$$

$z=\sqrt{2i+7} \iff z^2 = 2i+7 = \sqrt{53}\, e^{i\,\text{Arctan}(\frac{2}{7})}$

$\nearrow z_1 = (53)^{1/4} e^{i\,\frac{1}{2}\text{Arctan}(\frac{2}{7})}$ Principal root

$\to z_2 = -(53)^{1/4} e^{i\,(\frac{1}{2}\text{Arctan}(\frac{2}{7}))}$ other root.

Two roots of $z^2-(2i+7)=0$.

Solve $x^4 - x^3 + x^2 - x + 1 = 0$

$(x+1)(x^4 - x^3 + x^2 - x + 1) = 0$

$(x^5+1) = 0$

$\Rightarrow x^5 = -1 = e^{i\pi}$

$x = re^{i\theta}$

get 5 roots —

but exclude $x=-1$